AIAA JOURNAL
Vol. 32, No. 2, February 1994

Automatic Choice of Measurement Locations for Dynamic Testing

J. E. T. Penny,* M. I. Friswell,† and S. D. Garvey†
Aston University, Birmingham B4 7ET, England, United Kingdom

This paper examines the problem of choosing an optimum, or near-optimum, set of measurement locations for experimental modal testing and suggests criteria whereby the suitability of the chosen locations can be assessed. Two methods of coordinate selection are used: one based on Guyan reduction and the other on the Fisher information matrix. Each begins with a detailed finite element model of the structure being tested. Both procedures reduce this model by one degree of freedom at a time until the number of degrees of freedom in the reduced model equals the number of measurement locations required. The choice of the eliminated coordinates is generally automatic, and the coordinates of the reduced model are those used for modal testing. Five possible methods for assessing the suitability of the chosen measurement locations are considered, and examples are given of the application of these methods to simple structures.

Introduction

IN experimental modal testing the forcing and measurement locations chosen have a major influence on the quality of the results, and in extreme cases modes may be completely missed. For simple structures the experienced engineer is able to make a reasonable selection of coordinates, but for more complex structures the choice is often difficult.

Before a modal test is undertaken, it is necessary to decide how many measurement locations are required, where they should be located, and where the excitation should be applied. Clearly, it would be advantageous if a systematic procedure could be developed to choose the number and location of the measurement points. Such a procedure should be capable of giving an optimum or near-optimum selection. In addition to a systematic selection procedure, criteria must be established to assess the suitability of any set of measurement locations.

Selecting Measurement Locations

Two procedures are now considered for selecting measurement locations. The first method is based on classical Guyan reductions[1] and the second, developed by Kammer,[2–4] uses the Fisher information matrix and is called the effective independence distribution vector method. The latter method has been extended to the problem of selecting actuator locations, as well as measurement locations, to identify the modal parameters of a structure.[5]

Guyan Reduction

The purpose of the Guyan reduction, as originally conceived,[1] was to reduce the number of degrees of freedom in a large finite element (FE) model to make the solution of the resulting eigenvalue problem more manageable. The reduction is achieved by eliminating those coordinates for which the inertia forces are negligible compared with the elastic forces. This leads to a set of constraint equations relating the master coordinates (to be maintained) and the slave coordinates (to be removed). Introducing this constraint equation into the expressions for the kinetic and strain energies of the system leads to reduced mass and stiffness matrices in terms of the master coordinates.

The question arises as to how the master coordinates are to be selected. Here the underlying assumption in Guyan reduction must be borne in mind, that at slave coordinates the inertia forces are negligible compared with the elastic forces. Thus the slaves must be chosen where the inertia is low and the stiffness is high so that the mass is well connected to the structure. Conversely, the master coordinates are chosen where the inertia is high and the stiffness is low. This process can be automated[6] by examining the ratio k_{ii}/m_{ii} for the ith coordinate. If k_{ii}/m_{ii} is small, then there are significant inertia effects associated with this coordinate, and thus it should be retained as a master; if k_{ii}/m_{ii} is large, then the ith coordinate should be chosen as a slave and removed.

The slave coordinates are not chosen according to the previous rule *en bloc* but rather are chosen and removed one at a time. There are two advantages to this procedure. Firstly, at each stage the effect of each coordinate removed is redistributed to all of the remaining coordinates, so that the next reduction will remove the coordinates with the highest k_{ii}/m_{ii} ratio in the reduced mass and stiffness matrices. Secondly, there is a very simple algorithm for performing this sequential process of coordinate selection and removal.[6]

So far we have described the application of the Guyan reduction to large FE models for the purpose of generating a reduced model that accurately maintains the characteristics of the original model at the lower frequencies. In many respects the criterion for choosing the measurement locations in a large system is the same: we want to measure the lower frequency modes accurately. Thus it is reasonable to postulate that the master coordinates of an FE model can also serve as the measurement locations for modal testing. In each case it is reasonable to argue that the eliminated coordinates should be well connected into the structure.

In practice we determine the measurement locations as follows. We will begin with an FE model that will have many more coordinates than could be measured realistically. As a first step, before the automatic selection procedure begins, coordinates in the FE model that cannot readily serve as measurement locations are removed. These normally include rotational coordinates and coordinates that are not accessible. The automatic selection procedure is then used to reduce the number of master coordinates in the FE model to the number required for measurement purposes. Note that at each stage of the reduction process a reduced mass and stiffness matrix is generated. At the end of the process these matrices are generally not required.

Effective Independence

The objective of this method is to select measurement locations that make the mode shapes of interest as linearly independent as possible while retaining as much information as possible about the selected modal responses in the measurement data. The measurement location problem is approached from estimation theory.

As with Guyan reduction, we begin with an FE model of the system. The first step is to remove all coordinates that cannot be

Received March 13, 1993; revision received July 27, 1993; accepted for publication July 27, 1993.

*Senior Lecturer, Department of Mechanical and Electrical Engineering.
†Lecturer, Department of Mechanical and Electrical Engineering.

measured. Kammer[2–4] goes further than this and allows the removal of coordinates that are judged to be of little significance. The effective independence procedure is now begun by forming the Fisher information matrix $\boldsymbol{A}$, given by

$$\boldsymbol{A} = \boldsymbol{U}_m^T \boldsymbol{U}_m \tag{1}$$

where $\boldsymbol{U}_m$ is the reduced modal matrix. The modal matrix is reduced because it contains only the modes of interest, and these are only specified at the coordinates currently selected. The matrix **E** is then formed as follows:

$$\boldsymbol{E} = \boldsymbol{U}_m \boldsymbol{A}^{-1} \boldsymbol{U}_m^T \tag{2}$$

Matrix $\boldsymbol{E}$ is an idempotent matrix with the property that its trace equals its rank. Thus terms on the diagonal of $\boldsymbol{E}$ represent the fractional contribution of each measurement location to the rank of $\boldsymbol{E}$ and hence to the independence of the chosen modes. Matrix $\boldsymbol{E}$ will only be full rank if all of the modes of interest are linearly independent. The selection procedure is to examine the elements of the diagonal of $\boldsymbol{E}$. Since the smallest element relates to the coordinate that contributes least to the independence of the chosen modes, this degree of freedom is removed. The matrix $\boldsymbol{E}$ is then recomputed, and the process is repeated, removing coordinates iteratively. When the process is stopped, the remaining coordinates serve as the measurement locations. Kammer[2] states that this procedure eliminates degrees of freedom in a suboptimal manner to arrive at a set of measurement locations that will give the best chance of accurately identifying the experimental modes. The diagonal of $\boldsymbol{E}$ is called the effective independence distribution vector $\boldsymbol{E}_D$. We will henceforth refer to this method as the EIDV method.

Assessing the Suitability of Measurement Locations

It is highly desirable that a criterion be established to assess the quality of any set of chosen measurement locations. Here five possible criteria are investigated. They are not minimized directly because of the computational effort required.

Modal Assurance Criterion

In any modal test the mode shape data are only really useful when the measured modes can be distinguished from each other. That is, the mode shape vectors should be linearly independent. This is particularly important where the test results are to validate or update the FE model. Possibly the easiest way to check the linear dependence of the mode shapes is to use the modal assurance criterion (MAC).[7] Often the MAC is used to check the correlation between the measurement and analytical mode shapes. Here only the analytical mode shapes at the chosen measurement coordinates are used. Then the (i, j) entry of the MAC matrix is given by

$$\mathrm{MAC}_{ij} = \frac{\left[\boldsymbol{u}^{(i)T}\boldsymbol{u}^{(j)}\right]^2}{\boldsymbol{u}^{(i)T}\boldsymbol{u}^{(i)}\boldsymbol{u}^{(j)T}\boldsymbol{u}^{(j)}} \tag{3}$$

where $u^{(i)}$ is the ith mode shape, derived from the FE model. Only the selected degrees of freedom are listed. Note that the diagonal entries are unity. Usually the off-diagonal terms are not zero because the eigenvectors are orthogonal with respect to the mass and stiffness matrices and the MAC includes no such weightings. The MAC matrix may be computed using all of the degrees of freedom in the analytical model to provide a benchmark against which to judge the MAC matrix computed using mode shapes based on the chosen measurement locations. The larger the off-diagonal terms, the more dependent are the mode shape vectors. To obtain an average measure of the dependence of the vectors, the rms value of the off-diagonal entries in the MAC matrix is calculated.

Modified Modal Assurance Criterion

The notion of a modal assurance criterion (MAC) is derived from modal testing where the correlation between the measured and analytical mode shapes is required. In this application we have both the mode shape and the mass matrix derived from the FE model. Usually the mass matrix is not included because it would have to be derived from a reduced model based on the measured degrees of freedom only. Thus we can define a modified MAC:

$$\mathrm{ModMAC}_{ij} = \frac{\left[\boldsymbol{u}^{(i)T}\boldsymbol{M}\boldsymbol{u}^{(j)}\right]^2}{\boldsymbol{u}^{(i)T}\boldsymbol{M}\boldsymbol{u}^{(i)}\boldsymbol{u}^{(j)T}\boldsymbol{M}\boldsymbol{u}^{(j)}} \tag{4}$$

where $\boldsymbol{u}^{(i)}$ is the ith mode shape, derived from the FE model, and $\boldsymbol{M}$ is the reduced mass matrix. This reduced matrix could be derived from the full mass matrix by deleting rows and columns corresponding to the discarded coordinates. However, the orthogonality condition is more closely satisfied by a more accurately reduced mass matrix, such as that obtained by Guyan reduction. This would be a disadvantage if the measurement locations were chosen using the EIDV method as the reduced mass matrix is not available. The modified MAC has the possible advantage that when all of the degrees of freedom are included in the model, the off-diagonal terms are zero due to orthogonality. This may provide a better reference value with which to compare the selected measurement locations.

Singular Value Decomposition

A singular value decomposition (SVD) of the eigenvector matrix based on the measured degrees of freedom may be used to determine the suitability of the measurement locations chosen. The method simply evaluates the ratio of the largest to the smallest singular value of the eigenvector matrix. If this ratio is close to unity, then the choice of measurement locations is good. The larger the ratio, the worst the choice of locations. There are three ways of justifying the use of the SVD, namely, mode orthogonality, the condition of the mode expansion problem, and the observability of the modes. Each of these will now be considered in turn.

1. *Mode Orthogonality*

The desirability of this was discussed in the previous section. The MAC and the SVD are two ways of testing for orthogonality. If the mode shape vectors are orthogonal (the MAC matrix is the identity matrix), then all of the singular values of the modal matrix are equal. If the vectors are linearly dependent, then at least one singular value is zero.

2. *Mode Expansion*

Usually the number of measurement coordinates is low compared with the number of degrees of freedom in the FE model. To compare the eigenvectors from the FE analysis with the measured mode shapes requires the analytical eigenvectors to be reduced or the measured mode shapes to be expanded. O'Callahan et al.[8] derived an expression to expand a vector of responses at the measured coordinates into a vector of responses at the analytical coordinates. Thus

$$\boldsymbol{x}_a = \boldsymbol{U}_a \boldsymbol{U}_m^g \boldsymbol{x}_m \tag{5}$$

where, in the context of this paper, $\boldsymbol{x}_m$ is the vector of responses at the measured coordinates, $\boldsymbol{x}_a$ is the vector of estimated responses at all of the degrees of freedom, $\boldsymbol{U}_a$ is the matrix of analytical eigenvectors, $\boldsymbol{U}_m$ is the matrix of analytical eigenvectors at the measured coordinates, and the superscript g denotes the generalized inverse. In the case where the number of modes of interest is less than the number of measurement locations, $\boldsymbol{U}_m^g$ is given by

$$\boldsymbol{U}_m^g = \left(\boldsymbol{U}_m^T \boldsymbol{U}_m\right)^{-1} \boldsymbol{U}_m^T \tag{6}$$

The condition of the generalized inverse in Eq. (5) may be assessed by the ratio of the smallest to the largest singular values of the matrix of measured mode shapes. Thus the numerical condition of the mode expansion problem is determined by the SVD of the modal matrix.

3. Observability of the Modes of Interest

The basis of this argument is a little more involved and will not be considered in detail. The analytical equations of motion are transformed into modal coordinates. The observability matrix may then be formed. The rank of the observability matrix is determined by the rank of the matrix of eigenvectors based on the chosen measurement locations. A further treatment of the use of observability is given by Inman.[9]

Measured Energy per Mode

The kinetic energy of a system T is

$$T = \tfrac{1}{2}\dot{q}^T M \dot{q} = \tfrac{1}{2}\dot{p}^T U^T M U \dot{p} \tag{7}$$

where q is a vector of generalized coordinates, $\dot{q}$ denotes the derivative of q with respect to time, U is the full modal matrix, and p is the vector of principal coordinates. The principal coordinates are related to the generalized coordinates by $q = Up$. Considering now the kinetic energy of mode i only, we have

$$T_i = \tfrac{1}{2}\omega_i^2 p_i u^{(i)T} M u^{(i)} p_i \quad \text{or} \quad T_i = \alpha_i u^{(i)T} M u^{(i)} \tag{8}$$

where $u^{(i)}$ is the ith eigenvector, and p_i is the ith principal coordinate. If $u^{(i)}$ is complete and normalized, then $u^{(i)T} M u^{(i)} = 1$. However, since we are only measuring certain selected locations, we will divide $u^{(i)}$ and M into master coordinates m (i.e., measurement locations) and removed coordinates r. Thus

$$T_i = \alpha_i \begin{bmatrix} u_m^{(i)T} & u_r^{(i)T} \end{bmatrix} \begin{bmatrix} M_{mm} & M_{mr} \\ M_{rm} & M_{rr} \end{bmatrix} \begin{bmatrix} u_m^{(i)} \\ u_r^{(i)} \end{bmatrix} \tag{9}$$

or

$$T_i = \alpha_i \Big[u_m^{(i)T} M_{mm} u_m^{(i)} + u_m^{(i)T} M_{mr} u_r^{(i)} + (u_m^{(i)T} M_{mr} u_r^{(i)})^T + u_r^{(i)T} M_{rr} u_r^{(i)} \Big]$$

If $u^{(i)}$ is normalised with respect to the mass matrix, the sum of the terms in the square brackets is unity. Only $u_m^{(i)}$ is measured so that only the energy contained in the first term in the square brackets is measured. Comparing this term with unity is a measure of how well we are measuring the kinetic energy of the system. In fact, it is possible for the first term to exceed unity in some circumstances. This arises because, although the first and last terms in the square brackets must be positive, the middle two terms may be negative. Hence

$$u_m^{(i)T} M_{mr} u_r^{(i)} + \left[u_m^{(i)T} M_{mr} u_r^{(i)} \right]^T + u_r^{(i)T} M_{rr} u_r^{(i)} \tag{10}$$

may be negative, making the first term greater than one. It will be seen in the results that this does not happen often, but it does cast some doubt on the significance of the estimation.

Fisher Information Matrix

The determinant of the Fisher information matrix (FIM) indicates the amount of information in the data retained at the reduced set of coordinates.[2–4] Thus we wish to maintain a high value for this determinant so that the FIM retains as much information as possible. One disadvantage of the FIM is that the value of its determinant does not lie in a fixed range, and so it is difficult to judge from this value the quality of a selected set of measurement locations. The determinant of the FIM could be normalized by using the determinant of the FIM with all of the coordinates present, but even this normalized value would not lie in a fixed range. It can be shown that the condition number of the FIM is the square of the ratio of the largest to the smallest singular value of the eigenvector matrix.

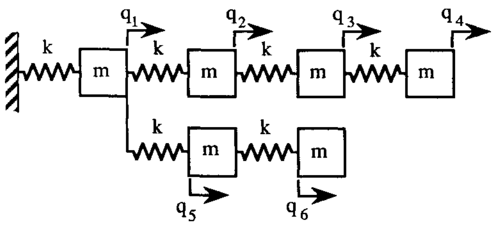


Fig. 1 Six degree-of-freedom system.

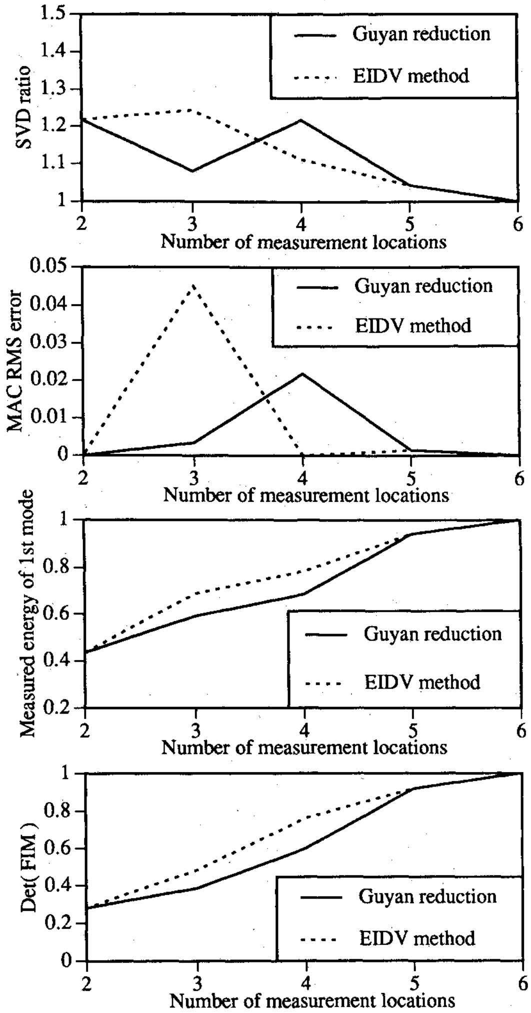


Fig. 2 Effect of the number of measurement locations and method of selection on various criteria (six degree-of-freedom system).

Examples of Measurement Location Selection

In Tables 1–9, Guyan 6 and EIDV 6 denotes that six measurement locations are selected by Guyan reduction and by the EIDV method, respectively. In the tables the MAC and modified MAC rms errors are the rms value of the off-diagonal elements in the respective MAC and modified MAC matrices. These rms errors are given in the tables to three decimal places because they must lie in the range 0–1. The modified MAC is computed using the Guyan reduced mass matrix. The SVD ratio is the ratio of the largest to the smallest singular value of the eigenvector matrix.

Example 1

We will begin by considering the trivial problem of selecting measurement locations for the six degree-of-freedom discrete system shown in Fig. 1. Assuming that we wish to identify only the first two modes of vibration, we will consider making measurements at two, three, four, five, and six locations. The first result of

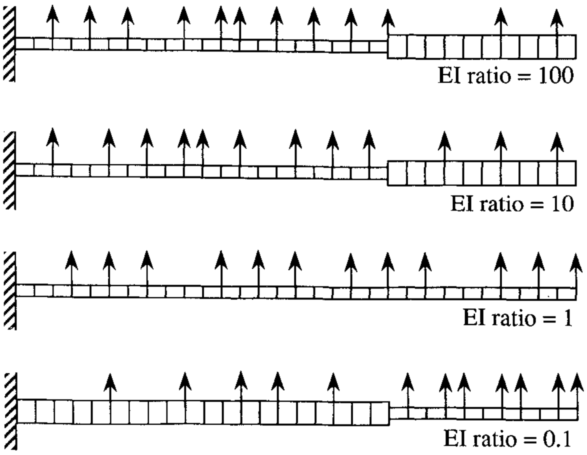


Fig. 3 Use of Guyan reduction to select 12 transducer locations for 4 cantilever beams (arrows indicate selected locations).

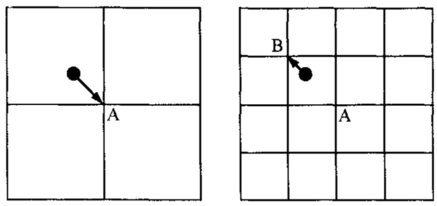


Fig. 4 Effect of mesh size on selected transducer location.

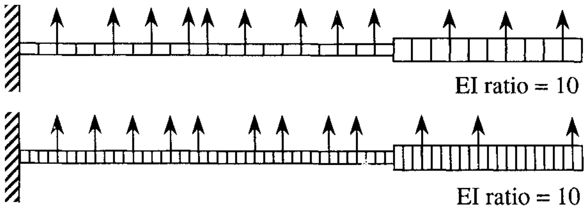


Fig. 5 Effect of doubling the number of degrees of freedom on the position of 12 transducer locations selected by Guyan reduction (arrows indicate selected locations).

interest is that even in this simple example Guyan reduction and the EIDV method choose a different set of locations. Starting with all six coordinates, Guyan reduction removes coordinates in the sequence 1, 3, 5, and 2 to leave coordinates 4 and 6. The EIDV method similarly begins by removing 1, but then follows with the sequence 2, 5, and 3 to again leave 4 and 6. Figure 2 shows the variation of the SVD ratio, the MAC rms error, the determinant of the FIM, and the measured energy of the first mode, as the number of measurement locations is varied from 2 to 6. The measured energy of the first mode is plotted because for all cases it is slightly smaller than the measured energy of the second mode. Only the measured energy of the first mode and the determinant of the FIM vary monotonically with the number of coordinates selected by the two methods. These measures will always increase when more coordinates are added to an existing set, and it could therefore be argued that they should be divided by the number of measurement locations chosen. The MAC rms error does not correlate well with the number of measurement locations, probably because in this example the rms error is very small.

Example 2

The cantilever beam shown in Fig. 3 was modeled with 30 elements and constrained to vibrate in one plane only. Axial extensions are ignored, giving the model 60 degrees of freedom. The outer part of the beam has an EI value that is different from the EI value of the inner part, where EI is the flexural rigidity. Let us assume that we wish to measure the first six elastic modes of vibration. All rotational degrees of freedom were removed before either the Guyan or EIDV procedures were used. Figure 3 shows the influence of α, the ratio of the EI of the outer section to the EI of the inner section of the beam when using Guyan reduction to select 12 measurement locations. When $\alpha = 100$, the outer section of the beam is so stiff that even when vibrating in the sixth mode there is virtually no bending in this section of the beam. As a consequence of this, both the Guyan and EIDV procedures place only two measurement locations in the outer section. As α is successively reduced, the measurement locations migrate toward the outer part of the beam, as we would expect. This is summarized in Table 1, which shows the mean position of a set of measurement locations as a percentage of the beam length for various values of α. In all cases the mean position of the transducers moves toward the free end of the cantilever as α decreases.

If there is an optimum distribution of measurement locations, then the selection procedure should place each measurement at the nearest coordinate in the model to the optimum location. Under these conditions changing the position of the coordinates may make small adjustments to the locations chosen for the transducers, but the locations should be those closest to the optimum. This is illustrated diagrammatically in Fig. 4. Here the nearest node to the "optimum" location in the coarse mesh is node A, but in the finer mesh it is no longer node A but node B. Figure 5 shows the effect of doubling the number of coordinates in the cantilever model from 60 to 120. Although the measurement locations chosen by Guyan reduction have not changed significantly, some have moved by more than one small element of the beam, indicating that the choice is not optimal and is, to some extent, dependent on the discretization chosen. This also applies to the selection made by the EIDV method. We will return to this problem in Example 4.

Tables 2 and 3 show numerical values for the various criteria proposed for assessing the suitability of the chosen measurement locations when $\alpha = 10$. Table 2 shows that the determinant of the

Table 1 Mean position of the chosen locations as a percentage of its length for the cantilever beam example

α	Guyan 6, %	Guyan 12, %	EIDV 6, %	EIDV 12, %
100	48	46	50	49
10	58	48	53	52
1	62	55	58	60
0.1	70	66	68	66

Table 2 Coordinate location criteria for the cantilever beam example, $\alpha = 10$

	Modified MAC rms error	MAC rms error	SVD ratio	Determinant of FIM
Guyan 6	0.101	0.095	4.66	0.027
Guyan 12	0.000	0.016	1.46	6.53
Guyan 18	0.000	0.021	1.54	81.0
EIDV 6	0.006	0.052	2.03	0.367
EIDV 12	0.000	0.037	1.84	19.8
EIDV 18	0.000	0.014	1.46	178
All vertical	0.000	0.002	1.13	1750

Table 3 Energy per mode for the cantilever beam example, $\alpha = 10$ (the mode with the lowest energy is underlined)

	Mode					
Energy/Mode	1	2	3	4	5	6
Guyan 6	0.178	0.202	0.170	0.195	0.114	0.152
Guyan 12	0.267	0.348	0.385	0.282	0.354	0.204
Guyan 18	0.365	0.621	0.638	0.614	0.552	0.622
EIDV 6	0.122	0.170	0.155	0.155	0.200	0.263
EIDV 12	0.328	0.428	0.405	0.409	0.450	0.528
EIDV 18	0.475	0.657	0.688	0.604	0.596	0.746
All vertical	1.000	0.998	0.993	0.983	0.971	0.961

FIM increases monotonically as the number of measurement locations is increased, whether they are selected using Guyan reduction or the EIDV method. The modified MAC quickly reduces to zero as the number of measurement locations increases. Table 3 shows the measured energy per mode for each of the six modes to be measured. In every case except one, the mode with the lowest measured energy (underlined in the table) is the first. Here the EIDV method is marginally better than Guyan reduction in selecting measurement locations that maximize the minimum measured energy in a mode. In Tables 2 and 3 the "All vertical" case uses all 30 vertical coordinates as measurement locations. As might be expected, all of the criteria assessed the 30 locations as a good selection for measurement.

Example 3

Figures 6 and 7 show a plane frame modeled with 96 degrees of freedom, and it is assumed that we wish to measure the first six modes of vibration. The diagrams show the location of 6, 12, and 18 measurement locations when chosen by Guyan reduction (Fig. 6) and the EIDV method (Fig. 7). Table 4 shows that as the number of measurement locations is increased, using either Guyan reduction or the EIDV method, all criteria show an improvement. Table 5 shows that, irrespective of the number of measurement locations and the method used to choose them, the first mode has the lowest measured energy. Figure 8 illustrates how six measurement locations might be chosen by an inexperienced person. This selection of measurement locations is referred to as "Corners" in Tables 4 and 5, and it is seen that by any criterion this is a very poor choice! Tables 4 and 5 also show the values of the various criteria if all 64 horizontal and vertical coordinates are used as measurement locations and if all 96 coordinates (including rotations) are

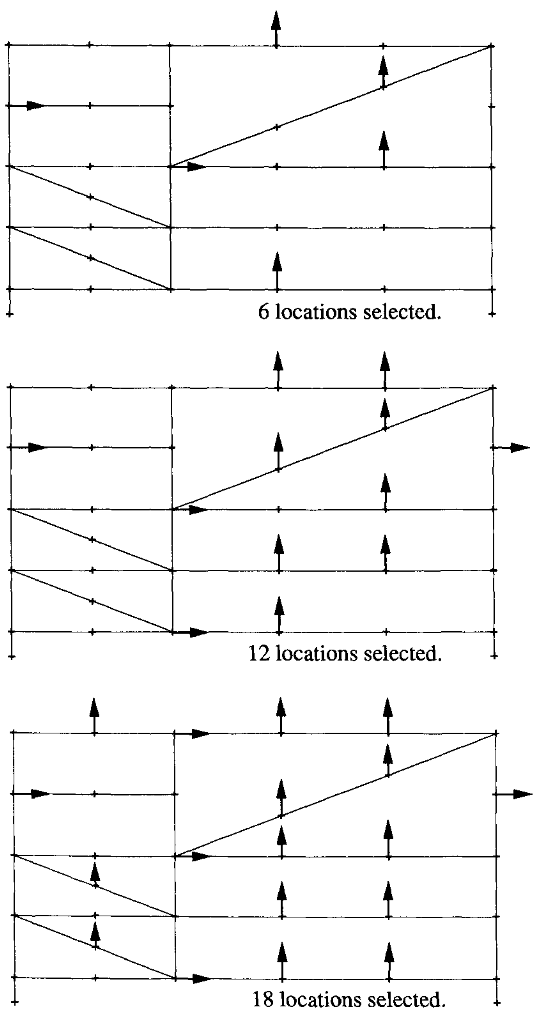


Fig. 6 Measurement locations on a plane frame chosen by Guyan reduction.

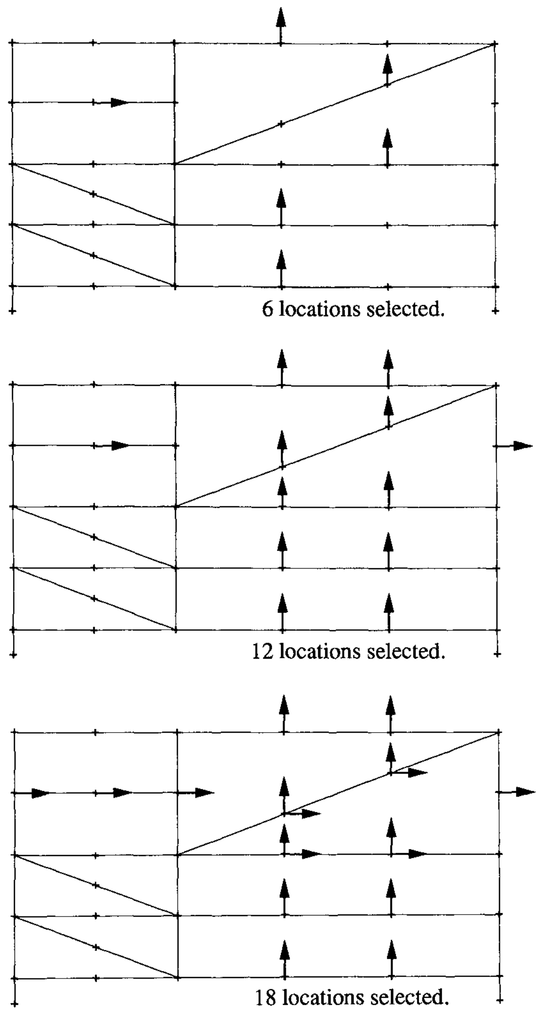


Fig. 7 Measurement locations on a plane frame chosen by the EIDV method.

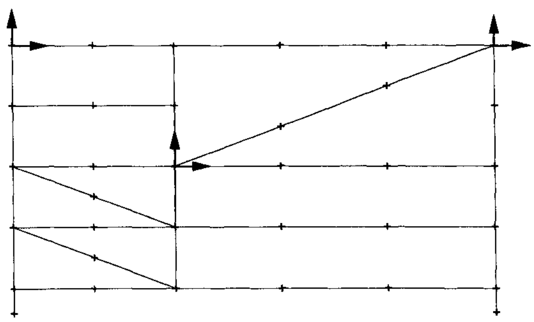


Fig. 8 Poor selection of six measurement locations on a plane frame.

used. In contrast to "Corners," these are shown to be an excellent, if impractical, choice.

Example 4

Figure 9 shows the distribution of 6 and 12 selected measurement locations on a plate, clamped along two edges and with a slot in one free edge. The plate is modeled with 90 degrees of freedom. Again it is assumed that we wish to measure the first six modes of vibration. Tables 6 and 7 show values of the assessment criteria when 6, 12, and 18 measurement locations are chosen by Guyan reduction and the EIDV method. Figure 10 shows the distribution of 6 and 12 measurement locations on the plate when it is modeled

Table 4 Coordinate location criteria for the plane frame example

	Modified MAC rms error	MAC rms error	SVD ratio	Determinant of FIM
Guyan 6	0.199	0.213	11.1	100
Guyan 12	0.002	0.043	3.00	1.56e+05
Guyan 18	0.000	0.028	2.67	6.60e+05
EIDV 6	0.024	0.042	3.61	5.84e+03
EIDV 12	0.010	0.061	3.60	3.48e+05
EIDV 18	0.000	0.037	2.00	2.00e+06
Corners	0.614	0.618	1160	0
Coordinates				
64	0.000	0.001	1.20	6.97e+06
96	0.000	0.012	3.19	5.00e+17

Table 5 Energy per mode for the plane frame example (the mode with the lowest energy is underlined)

	Mode					
Energy/Mode	1	2	3	4	5	6
Guyan 6	0.123	0.258	0.281	0.298	0.281	0.132
Guyan 12	0.187	0.539	0.669	0.359	0.650	0.713
Guyan 18	0.210	0.543	0.790	0.678	0.866	0.870
EIDV 6	0.040	0.242	0.283	0.294	0.381	0.368
EIDV 12	0.104	0.520	0.775	0.652	0.862	0.864
EIDV 18	0.334	0.799	0.857	0.687	0.869	0.878
Corners	0.114	0.019	0.001	0.023	0.002	0.001
Coordinates						
64	0.979	0.889	0.863	0.912	0.889	0.895
64	1.000	1.000	1.000	1.000	1.000	1.000

Table 6 Coordinate location criteria for the clamped plate example with 90 degrees of freedom

	Modified MAC rms error	MAC rms error	SVD ratio	Determinant of FIM
Guyan 6	0.046	0.200	5.18	4.59e+04
Guyan 12	0.002	0.066	2.10	5.62e+06
Guyan 18	0.000	0.083	2.18	2.53e+07
EIDV 6	0.009	0.129	3.42	1.45e+06
EIDV 12	0.000	0.098	2.80	3.71e+07
EIDV 18	0.000	0.062	2.18	1.93e+08
All vertical	0.000	0.048	1.90	4.57e+08
All coordinates	0.000	0.091	5.25	2.31e+20

Table 7 Energy per mode for the clamped plate example with 90 degrees of freedom (the mode with the lowest energy is underlined

	Mode					
Energy/Mode	1	2	3	4	5	6
Guyan 6	0.157	0.182	0.189	0.264	0.132	0.166
Guyan 12	0.372	0.360	0.456	0.315	0.321	0.324
Guyan 18	0.851	0.551	0.556	0.475	0.446	0.370
EIDV 6	0.124	0.241	0.154	0.283	0.214	0.285
EIDV 12	0.378	0.417	0.520	0.608	0.462	0.585
EIDV 18	0.749	0.931	0.736	0.696	0.657	0.660
All vertical	1.014	1.029	0.978	1.027	0.938	0.910
All coordinates	1.000	1.000	1.000	1.000	1.000	1.000

with 348 degrees of freedom. The corresponding data are given in Tables 8 and 9. Comparing Figs. 9 and 10, it is seen that, although the pattern of measurement locations is similar in each case, some locations have changed with the change in model by more than the minimum mesh size. This indicates that both methods of choosing measurement locations are dependent, to some degree, on the discretization of the structure. Corresponding data in Table 6 (based

Table 8 Coordinate location criteria for the clamped plate example with 348 degrees of freedom

	Modified MAC rms error	MAC rms error	SVD ratio	Determinant of FIM
Guyan 6	0.048	0.129	3.19	3.94e+04
Guyan 12	0.000	0.057	1.90	2.98e+06
Guyan 18	0.000	0.045	1.70	6.57e+06
EIDV 6	0.013	0.125	3.51	1.79e+06
EIDV 12	0.012	0.104	3.04	6.77e+07
EIDV 18	0.001	0.095	2.89	5.45e+08

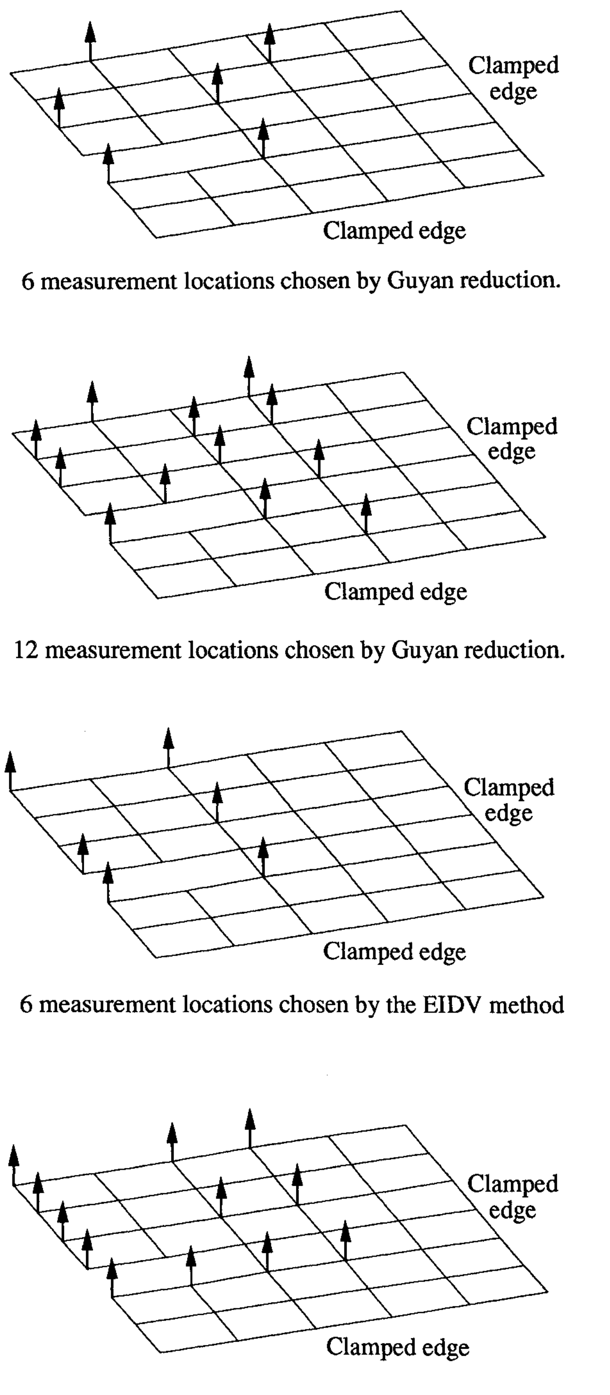
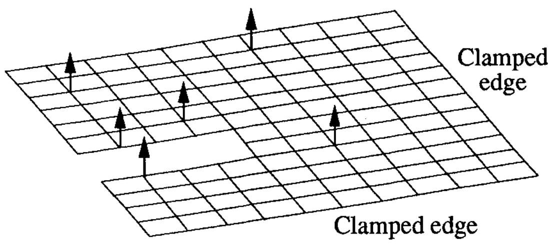
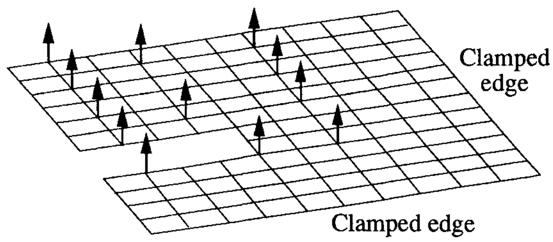


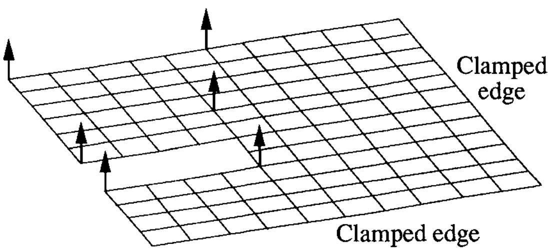
Fig. 9 Chosen measurement locations on a clamped plate with a slot, modeled with 90 degrees of freedom.



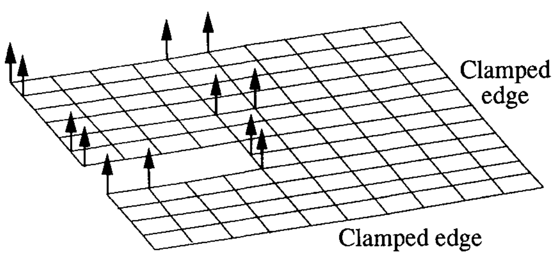
6 measurement locations chosen by Guyan reduction.



12 measurement locations chosen by Guyan reduction.



6 measurement locations chosen by the EIDV method



12 measurement locations chosen by the EIDV method

Fig. 10 Chosen measurement locations on a clamped plate with a slot, modeled with 348 degrees of freedom.

Table 9 Energy per mode for the clamped plate example with 348 degrees of freedom (the mode with the lowest energy is underlined)

	Mode					
Energy/Mode	1	2	3	4	5	6
Guyan 6	0.050	0.063	0.041	0.081	0.064	0.052
Guyan 12	0.113	0.125	0.086	0.107	0.093	0.081
Guyan 18	0.126	0.128	0.096	0.135	0.110	0.104
EIDV 6	0.028	0.055	0.042	0.071	0.062	0.080
EIDV 12	0.093	0.142	0.125	0.193	0.147	0.163
EIDV 18	0.148	0.246	0.178	0.304	0.256	0.287

on the 90-DOF model) and Table 8 (based on the 348-DOF model) do not differ greatly. However, the measured energy per mode for the 384-DOF model (Table 9) is much less than the measured energy per mode for the 90-DOF model (Table 7). This is because, in the 348-DOF model, we are measuring at a much smaller percentage of the coordinates than in the 90-DOF model. Thus we cannot directly compare the energy measured per mode when using different models of a structure.

Discussion and Conclusions

The following questions arise from this paper.

Which, if any, of the criteria used to assess the choice of measurement locations is correct? This is an important issue since if the criteria are wrong then any conclusions we may draw concerning the quality of a set of measurement locations is wrong! When we compare the information contained in Tables 2–9, we see that, although there is a broad agreement between the criteria, there are many detailed discrepancies. For example, in Table 6, an increase in the number of measurement locations chosen by Guyan reduction from 12 to 18 results in the modified MAC rms error, the MAC rms error, and the SVD ratio becoming slightly worse; only the determinant of the FIM improves. However, it is difficult to assess how good a set of measurement locations are in an absolute sense from the determinant of the FIM. For example, selecting 18 measurement locations by the EIDV method makes the determinant of the FIM equal to 178 in example 2 and approximately 2×10^6 in example 3. In contrast, the SVD ratio for those two cases is 1.46 and 2.00, respectively. This compares with a minimum value of 1, indicating that all of the pairs of measured modes are linearly independent. Two aspects of this SVD criterion are worth noting. Firstly, the eigenvectors must all be scaled according to a consistent rule, and the SVD ratio is dependent on the rule used. Secondly, if two measured modes are identical, the SVD-based criterion judges them to be infinitely bad. This is in contrast to the MAC rms error. Here the significance of the unit value that would occur in one pair of off-diagonal terms would be masked when the rms value of the off-diagonal terms is taken. When all of the FE model coordinates are used, the SVD ratio and rms values of the off-diagonal terms in the MAC matrix are sometimes not the lowest values. This implies that the increased number of coordinates actually reduces the orthogonality of the corresponding vectors.

Overall the SVD ratio criterion appears to be the best. The energy per mode works well, but it can sometimes be greater than unity and it has difficulty comparing the performance with different numbers of locations. The determinant of the FIM increases with the number of measurement locations but cannot give an absolute assessment of the quality of a set of locations. The MAC matrix suffers from not having a suitable reference matrix against which to gauge improvements. The modified MAC has the identity matrix as a reference, but the rms value of the off-diagonal elements quickly goes to zero as the number of measurement locations increases.

Do the methods produce a reasonable choice of measurement locations? Based on the criteria given in this paper, both the Guyan reduction and the EIDV method give an acceptable selection in most cases but not necessarily the optimum set. The selected set of measurement locations depends to some extent on the discretization used, and different models will produce similar but not identical selections.

The EIDV method aims to optimize the linear independence of the modes and therefore might be expected to perform better than Guyan reduction when assessed by the MAC and SVD ratio criteria. This is not always the case and highlights the fact that the EIDV method is suboptimal because coordinates are eliminated sequentially. Other measurement location choices may produce vectors that are closer to being orthogonal.

Are there better ways of choosing measurement locations? To answer this question we must assume that a suitable criterion has been established to measure the quality of the choice of measurement locations. This paper has described five such criteria. The two methods used in this paper to select measurement locations do not guarantee an optimum selection based on these criteria but give reasonable results. Obviously there is scope to produce an optimum choice. For example, if a quality criterion could be established, then it could be maximized directly, perhaps using techniques of linear programming. The advantage of the Guyan reduction and EIDV method is that they are easy to implement and not too computer intensive.

References

[1]Guyan, R. J., "Reduction of Stiffness and Mass Matrices," *AIAA Journal*, Vol. 3, No. 2, 1965, p. 380.

[2]Kammer, D. C., "Sensor Placements for On-Orbit Modal Identification and Correlation of Large Space Structures," *Journal of Guidance, Control, and Dynamics*, Vol. 14, No. 2, 1991, pp. 251–259.

[3]Kammer, D. C., "Effect of Noise on Sensor Placement for On-Orbit Modal Identification and Correlation of Large Space Structures," *Journal of Dynamic Systems, Measurement and Control—Transactions of ASME*, Vol. 114, No. 3, 1992, pp. 436–443.

[4]Kammer, D. C., "Effect of Model Error on Sensor Placement for On-Orbit Modal Identification and Correlation of Large Space Structures," *Journal of Guidance, Control, and Dynamics*, Vol. 15, No. 2, 1992, pp. 334–341.

[5]Lim, T. W., "Actuator/Sensor Placement for Modal Parameter Identification of Flexible Structures," *Modal Analysis*, Vol. 8, No. 1, 1993, pp. 1–14.

[6]Henshell, R. D., and Ong, J. H., "Automatic Masters for Eigenvalue Economisation," *Earthquake Engineering and Structural Dynamics*, Vol. 3, No. 4, 1975, pp. 375–383.

[7]Allemang, R. J., and Brown, D. L., "A Correlation Coefficient for Modal Vector Analysis," *Proceedings of the 1st International Modal Analysis Conference* (Orlando, FL), Union College, Schenectady, NY, 1982, pp. 110–116.

[8]O'Callahan, J., Avitabile, P., and Riemer, R., "System Equivalent Reduction Expansion Process," *Proceedings of the 7th International Modal Analysis Conference* (Las Vegas, NV), Union College, Schenectady, NY, 1989, pp. 29–37.

[9]Inman, D. J., *Vibration with Control, Measurement and Stability*, Prentice-Hall, Englewood Cliffs, NJ, 1989.